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A new single surface integral equation for light scattering by circular dielectric cylinders

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Abstract

A new single surface integral equation is derived for light scattering by circular dielectric cylinders. Without adopting the concept of equivalent electric or magnetic surface currents, our formulation is directly derived from coupled-surface integral equations by the property of commutative matrices of Green functions. Further development by such matrix equations leads to only one unknown function for circular dielectric-coated cylinders. In addition, numerical simulations show that even applied to elliptic scatters our equation still gives reasonably good approximate solutions in the sub-wavelength limit.

For scattering problems in electrodynamics, integral approach, in contrast to differential methods, possesses the analytical characteristic of solution of Helmholtz equation with point source, namely, an integration of Green function. With the use of Green identity, the scattering field is obtained by the integral of the total field and its normal derivative on the enclosed surface of the object. In scatters of perfect conductor, the integrand reduces to only one variable, the normal derivative of field, since the field is zero on perfect conductor. So the integral becomes with one unknown of Neumann boundary condition. In dielectric homogeneous object, because none of the two variables vanishes; one single surface Green integral indeed cannot be solved with two boundary conditions. For finding these two unknown functions on the boundary, dual surface integral equations are indispensable when such integrals are numerically expressed in two sets of linear algebra equations [1,2]. For three-dimensional (3D) scattering of arbitrary-shaped body, the coupled vector integral equations require one to solve a set of unknown equivalent electric and magnetic surface currents [3–5]. The matching of boundary conditions between two media in scattering is accomplished by derivation of the fields from electric and magnetic potential via the corresponding equivalent currents. In two dimensional (2D) scattering, due to the advantage of decomposition of field into TE and TM components, the respective Helmholtz equation can be treated as scalar scattering problems and discussed separately; and the equivalent surface current is essentially equivalent to the normal derivative of field. The studies of 2D electromagnetic scattering by integral equation method are extensively found in literature [6–10]. Nevertheless, the double-loaded coupled integral equations seem unpleasant in attempting the solution by numerical scheme. The first effort to reduce these two integral equations into one in 2D problems was proposed by Maystre [9,10]. With the derivation from distribution theory, Maystre successfully expressed the boundary field and its normal derivative in terms of a single surface current function. Once the equivalent current is found, the fields on the boundary can be obtained through a conversion integral by substitution of the current function. Later, the same idea was applied and generalized to 3D scattering problems [11–13]. With all in common, the substitution of the real field with an equivalent surface current function is the core constituent in these formulations.

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In this communication, we derive a new single integral equation without the conception of equivalent principle for the cases of circular scattering in 2D scattering. That is, none of the surface current is necessary in our formulation. Our approach is based on the use of commutative matrices to simplify two coupled linear algebra equations into one. With the same technique applied to circular dielectric-coated cylinders, we successfully reduce a set of four integral equations into one single linear algebra equation. All these works with consistent numerical results are addressed as follows.

Let us consider the scattering from a circular dielectric cylinder as shown in Fig. 1. In spite of the existence of the analytical solution [4], the coupled-surface integral equations for TE polarized wave are,

$$\frac{1}{2}E_1(x) = E_{\mathrm{inc}}(x) + \oint_{\mathrm{S}} -G_1(x,x')\frac{\partial E_1(x')}{\partial n'} + \frac{\partial G_1(x,x')}{\partial n'}E_1(x,x')\mathrm{d}s', \tag{1}$$
$$\frac{1}{2}E_2(x) = \oint_{\mathrm{S}} G_2(x,x')\frac{\partial E_2(x')}{\partial n'} - \frac{\partial G_2(x,x')}{\partial n'}E_2(x,x')\mathrm{d}s',$$

where the subscripts 1 and 2 referred to the homogeneous dielectric media 1 and 2, E_{inc} the incident field, the normal derivative over n' pointing from medium 2 to 1, and the slash on S denotes the principle Cauchy integral. $G_{1,2}$ are the respective Green functions in the medium 1 and 2 here. With the continuity of the field and its normal derivative on the boundary, namely, $E_1 = E_2$ and $\partial_{n'}E_1 = \eta\partial_{n'}E_1$ ($\eta = 1$ for TE polarization and $\eta = \varepsilon_1/\varepsilon_2$ for TM wave H, ε is the electric permittivity; throughout this paper only TE wave with constant vacuum magnetic permeability is considered), Eq. (1) becomes the following two $N \times N$ matrix equations by "point matching" method:

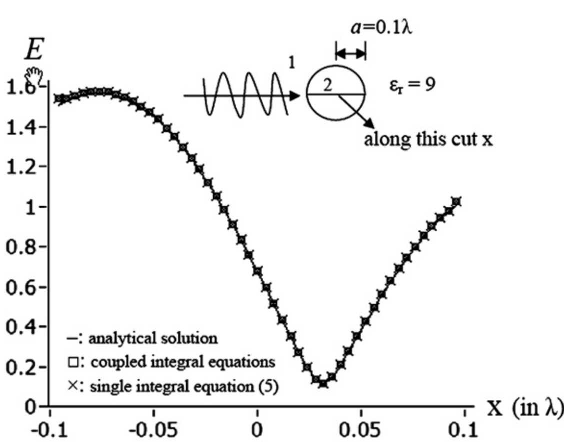

Fig. 1. The scattering field (normalized to incident TE wave) inside a dielectric circular cylinder along the cut of diameter, with radius $a = 0.1\lambda$ and relative electric permittivity $\varepsilon_r = 9$; the simple-line stands for analytical solution, square-line for solution from coupled integrals equation, and cross-line for solution from the single integral Eq. (5) developed here.

$$\frac{1}{2}[\mathbf{E}(x)] = -[\mathbf{G}_1(x,x')][\partial_{n'}\mathbf{E}(x')] + [\partial_{n'}\mathbb{G}_1(x,x')][\mathbf{E}(x')]\Delta s + [\mathbf{E}_{\mathrm{inc}}(x)],$$
$$\frac{1}{2}[\mathbf{E}(x)] = [\mathbf{G}_2(x,x')][\partial_{n'}\mathbf{E}(x')] - [\partial_{n'}\mathbf{G}_2(x,x')][\mathbf{E}(x')]\Delta s, \tag{2}$$

where we have chosen N partitions, with $x = x_1, \ldots, x_N$, on the boundary and the surface length element is Δs. The diagonal elements $\partial_{n'}\mathbf{G}_1(x',x') = \partial_{n'}\mathbf{G}_2(x',x') = 0$ are excluded from the matrices $\partial_{n'}\mathbf{G}_{1,2}$ as a fact of the principle Cauchy integral. In addition, the respective diagonal elements of $\mathbf{G}_{1,2}$ are of the same values as a result from the analytic infinitesimal integration of Hankel function on Δs [14]. Consider the symmetric Green function of its two variables, i.e., $G(x,x') = G(x',x)$, along with the integral path of circular cylinder, we found an important property of $\mathbf{G}_{1,2}$ and $\partial_{n'}\mathbf{G}_{1,2}$: they are indeed the *circulant* matrices, which satisfy the definition,

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_N \\ a_N & a_1 & a_2 & & a_{N-2} & a_{N-1} \\ a_{N-1} & a_N & a_1 & & & \vdots \\ \vdots & & & & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_N & a_1 \end{bmatrix}. \tag{3}$$

Such a matrix has its next row elements shifted in a circular order while keeping the diagonal elements the same values. A significant feature of circulant matrices is: for any two circulant matrices $\mathbf{A}$, $\mathbf{B}$ they obey the commutation rule, i.e., $\mathbf{AB} = \mathbf{BA}$ [15]. With this nice property, we can readily reduce the coupled equations in Eq. (2) to a single linear algebra equation by multiplying $\mathbf{G}_2$ and $\mathbf{G}_1$ to its upper and lower equations and add them up to eliminate the undesired $\partial_{n'}\mathbf{E}$ term. Thus, we have,

$$\frac{1}{2}[(\mathbf{G}_1 + \mathbf{G}_2)][\mathbf{E}] = \{[\mathbf{G}_2][\partial_{n'}\mathbf{G}_1] - [\mathbf{G}_1][\partial_{n'}\mathbf{G}_2]\}[\mathbf{E}]\Delta s + [\mathbf{G}_2][\mathbf{E}_{\mathrm{inc}}]. \tag{4}$$

Or, Eq. (4) can be recast into the integral form,

$$\oint_{S'}\oint_{S''}\left\{G_2(x,x')\frac{\partial G_1(x,x'')}{\partial n''} - G_1(x,x')\frac{\partial G_2(x,x'')}{\partial n''}\right\}E(x'')\mathrm{d}s''\mathrm{d}s' = \oint_{S'} G_2(x,x')E_{\mathrm{inc}}(x')\mathrm{d}s', \tag{5}$$

where we have included the diagonal elements with the values $+1/(2\Delta s_k)$ and $-1/(2\Delta s_k)$ into the matrices $\partial_{n''}\mathbf{G}_2$ and $\partial_{n''}\mathbf{G}_1$, respectively.

The verification of the validity of Eq. (4) or (5) is confirmed with numerical calculation. The results compared with coupled equations method and analytical solutions are shown in Fig. 1. We further apply the commutative property of Green matrices to dielectric-coated cylinders, where no analytical solution has been found. Before that, an extensive study with Eq. (5) for non-circular scattering has practical interest for applications. Since the integral paths are not perfectly circular, the corresponding Green function matrices are not circulant matrices. As a result, the Green matrices would violate the commutation rule, i.e., $\mathbf{AB} \neq \mathbf{BA}$, so that Eq. (5) no longer holds. However, if we choose a shape slightly deviated from circle, for example, an ellipse with small eccentricity, then $\mathbf{AB} \approx \mathbf{BA}$. Numerical simulation shows that, in the sub-wavelength limit (if the major axis of ellipse a is around or less than one tenth of the wavelength λ, i.e., $a \leqslant 0.1\lambda$), Eq. (5) is still a good approximation even at large eccentricity. These results are given in Fig. 2. We can see that the maximum error is reasonably small, only 2.13% at the eccentricity equal to 0.8.

Next, we use the non-trivial commutative property of circulant Green matrices to simplify the scattering problems of dielectric-coated cylinders. Consider the configuration illustrated in Fig. 3, a circular cylinder with refractive index n_3 is covered with a dielectric coating of refractive index n_2, illuminated in the medium with refractive index n_1. In the integral approach, it would require four unknown functions because we have two boundaries S_1 and S_2 separating the incident medium with coating and the coating with cylinder, respectively. In Maystre's work [16], an iterative relation between single jump functions, or the surfaces currents, on different layers was developed. Nevertheless, that formula is only valid for the case of vanished inner field, i.e., for cylinder considered here, the core material should be perfect conductor. Another similar treatment in operator form for non-vanished inner field still requires two integral equations [17]; to obtain single matrix equation, the inversion of the Green matrices in steps of reduction may cause numerical stability problems [18]. Here we use the same technique in deriving Eq. (4) to lead the reduction of four unknown functions: E_1 and $\partial_{n'}E_1$ on boundary S_1, and E_2 and $\partial_{n'}E_2$ on boundary S_2. The detailed derivation and the recast of integral form are skipped here. After a lengthy calculation, we arrive at the following matrix equation with only a single unknown function E_1.

$$\begin{aligned}
&[\{(\mathbf{G}_{221}\mathbf{G}_{212} - \mathbf{G}_{211}\mathbf{G}_{222})\mathbf{G}_{111}\partial\mathbf{G}_{212} \\
&\quad + \mathbf{G}_{111}\mathbf{G}_{212}(\mathbf{G}_{211}\partial_-\mathbf{G}_{222} - \mathbf{G}_{221}\partial\mathbf{G}_{212})\}\{(\mathbf{G}_{221}\mathbf{G}_{211} \\
&\quad - \mathbf{G}_{212}\mathbf{G}_{221})\mathbf{G}_{322}\partial\mathbf{G}_{221} + \mathbf{G}_{322}\mathbf{G}_{221}(\mathbf{G}_{212}\partial\mathbf{G}_{221} \\
&\quad - \mathbf{G}_{222}\partial_+\mathbf{G}_{211})\} + \{(\mathbf{G}_{221}\mathbf{G}_{211} - \mathbf{G}_{212}\mathbf{G}_{221}) \\
&\quad \times (\mathbf{G}_{222}\partial_+\mathbf{G}_{322} - \mathbf{G}_{322}\partial_-\mathbf{G}_{222}) + \mathbf{G}_{322}\mathbf{G}_{221}(\mathbf{G}_{222}\partial\mathbf{G}_{212} \\
&\quad - \mathbf{G}_{212}\partial_-\mathbf{G}_{222})\}\{(\mathbf{G}_{221}\mathbf{G}_{212} - \mathbf{G}_{211}\mathbf{G}_{222})(\mathbf{G}_{111}\partial_+\mathbf{G}_{211} \\
&\quad - \mathbf{G}_{211}\partial_-\mathbf{G}_{111}) + \mathbf{G}_{111}\mathbf{G}_{212}(\mathbf{G}_{211}\partial\mathbf{G}_{221} \\
&\quad - \mathbf{G}_{221}\partial_+\mathbf{G}_{211})\}][\mathbf{E}_1]\Delta s_1 \\
&\quad = [\{(\mathbf{G}_{221}\mathbf{G}_{211} - \mathbf{G}_{212}\mathbf{G}_{221})(\mathbf{G}_{222}\partial_+\mathbf{G}_{322} \\
&\qquad - \mathbf{G}_{322}\partial_-\mathbf{G}_{222}) + \mathbf{G}_{322}\mathbf{G}_{221}(\mathbf{G}_{222}\partial\mathbf{G}_{212} \\
&\qquad - \mathbf{G}_{212}\partial_-\mathbf{G}_{222})\}(\mathbf{G}_{221}\mathbf{G}_{212} - \mathbf{G}_{211}\mathbf{G}_{222})\mathbf{G}_{211}][\mathbf{E}_{\mathrm{inc}}], \qquad (6)
\end{aligned}$$

where the Green functions in media 1, 2, and 3 and their normal derivatives on S_1 and S_2 are denoted as $\mathbf{G}_{ijk} = \mathbf{G}_i(x_j, x'_k)$, $\partial\mathbf{G}_{ijk} = \partial\mathbf{G}_i(x_j, x'_k)/\partial n'_k$, with $i = 1, 2, 3$; $j, k = 1$ and 2, and $\partial_+\mathbf{G}_{ijk}$ and $\partial_-\mathbf{G}_{ijk}$ represent their diagonal elements with values of $+1/(2\Delta s_k)$ and $-1/(2\Delta s_k)$, respectively. The numerical results for scattering fields obtained from Eq. (6) are consistent with those from four coupled integral equations, as indicated in Fig. 3.

In conclusion, we have discovered the circulant Green functions and developed new single surface integral equations for 2D circular dielectric cylinders. Instead of using equivalent principle, our integrands contain the real fields only, and they are derived directly without any assumption. Therefore, no extra conversion is needed in our scheme to obtain the fields in comparison with the equivalent currents method. It works equally well for quasi-circular cylinders in the sub-wavelength limit. And we also succeed in reducing four integrals into one for cylinders with dielectric coat-

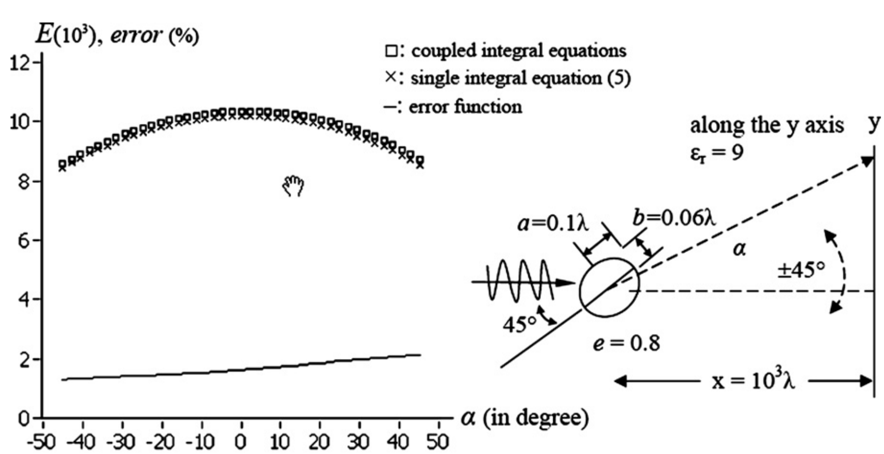

Fig. 2. The normalized scattering field (in units of 10^3) by an elliptic dielectric cylinder with eccentricity $e = 0.8$, long major axis $a = 0.1\lambda$, short major axis $b = 0.06\lambda$ and relative electric permittivity $\varepsilon_r = 9$, at 45° incidence angle respect to the long major axis. The fields are calculated at a horizontal distance $x = 10^3\lambda$ and vertical distributions along the y-axis at angle α of range ±45° with respect to x-axis from the center of cylinder. The square-line is the solution from coupled integrals equation, cross-line from the single integral Eq. (5) developed here, and the simple-line is the error function for cross-line relative to square-line defined as error = |cross-square|/|square| in percentage.

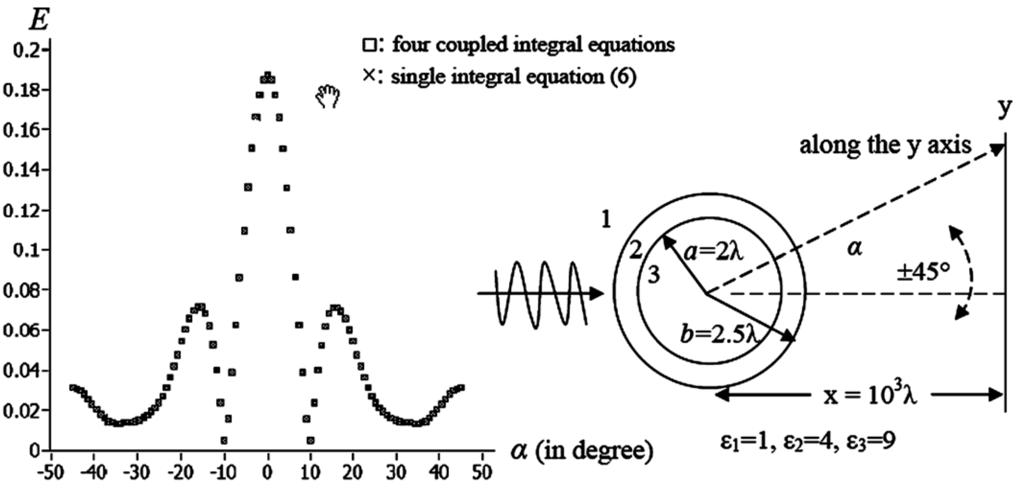

Fig. 3. The normalized scattering fields by a circular dielectric-coated cylinder, calculated at the same distances as in Fig. 2. The refractive indexes of the incident medium, coating, and the core are $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, with corresponding relative electric permittivity $\varepsilon_1 = 1$, $\varepsilon_2 = 4$, and $\varepsilon_3 = 9$; the inner and outer radii of the cylinder are $a = 2\lambda$, $b = 2.5\lambda$, respectively. The square-line is the solution from four coupled integrals equation with two boundary conditions, and the cross-line is the solution from the single integral Eq. (6) developed here.

ing. In future applications, our approach may further be implemented to multi-concentric circular layered media, which would reduce at least half of the unknowns on the integral boundaries. Moreover, our results can be specifically extended to 3D spherical scattering. For the sphere object only, the field can be generally defined into TE and TM polarizations due to the uniqueness of normal vector on each surface element [19]. Also the circulant Green matrices still hold because of the permutation of surface elements in 3D Green function on the sphere. With the same spirit, the reducing procedure in above coated cylinders can apply to spherically layered media as well.

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References

[1] A. Wirgin, Rev. Cethedec 5 (1968) 131.
[2] A.R. Neureuther, Alta Freq. 38 (1968) 282.
[3] A.J. Poggio, E.K. Miller, Integral equations solutions of three-dimensional scattering problems, in: Computer Techniques for Electromagnetics, Pergamon, Oxford, 1973.
[4] R. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill, New York, 1961.
[5] A. Peterson, S. Ray, R. Mitta, Computational Methods for Electromagnetics, IEEE Press, New York, 1998.
[6] V. Solodukhov, E. Vasilev, Soviet Phys.-Tech. Phys. 15 (1970) 32.
[7] N. Morita, Trans. Inst. Elec. Commun. Eng. Japan 57-B (1974) 2.
[8] N. Morita, N. Kumagai, J. Mautz, Integral Equation Methods for Electromagnetics, Artech House, Norwood, 1990.
[9] D. Maystre, P. Vincent, Opt. Commun. 5 (1972) 327.
[10] D. Maystre, Integral Methods, in: Electromagnetic Theory of Gratings, Spring-Verlag, Berlin, 1980.
[11] E. Marx, J. Math. Phys. 23 (1982) 1057.
[12] A.W. Glisson, IEEE Trans. Antennas Propoga. AP-32 (1984) 173.
[13] M.S. Yeung, Micro. Opt. Tech. Lett. 32 (2002) 383.
[14] N.I. Muskhelishvili, Singular Integral Equations, Wolters-Noordhoff, Groningen, 1958.
[15] P. Lancaster, Theory of Matrices, Academic Press, New York, 1969.
[16] D. Maystre, J. Opt. Soc. Am. 68 (1978) 490.
[17] A. Pomp, J. Mod. Opt. 38 (1991) 109.
[18] L.I. Goray, S. Yu. Sadov, in: Diffractive Optics and Micro-Optics, OSA Trends in Optics and Photonics 75 (2002) 365.
[19] B. Ya. Zeldovich, (private communication).